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Spontaneously stochastic solutions in turbulence models

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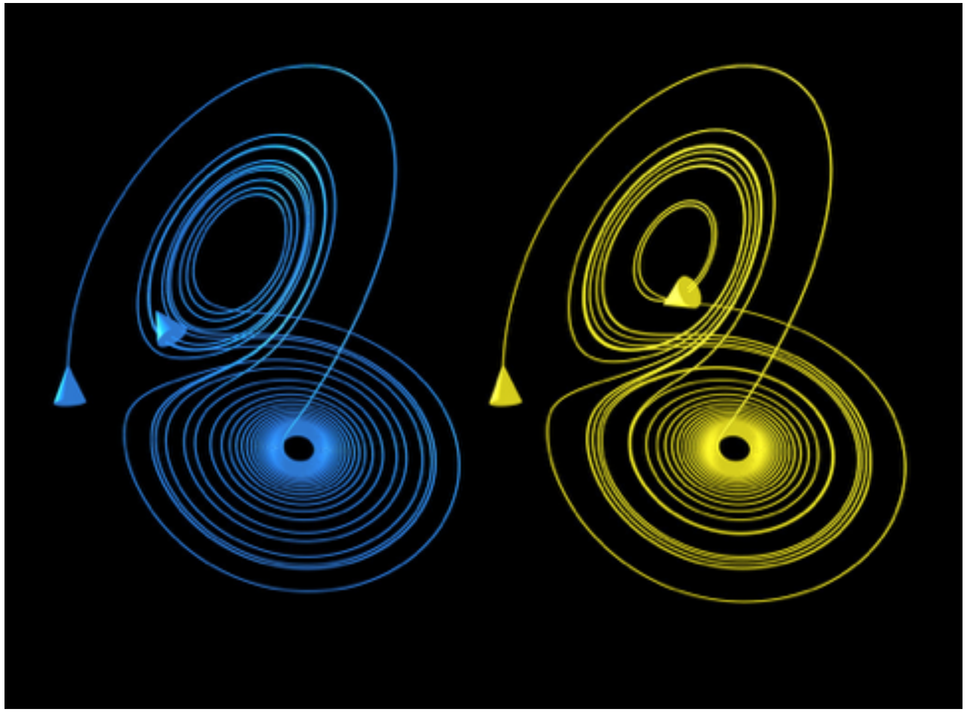
SPONTANEOUSLY STOCHASTIC SOLUTIONS IN TURBULENCE MODELS

Alexei Mailybaev (IMPA, Rio de Janeiro)

Introduction

Butterfly effect



Deterministic finite-dimensional chaos:

$$T \propto \frac{\log(1/\Delta x)}{\lambda_{\max}}$$

T is the separation time;

Δx is the initial observation error;

$\lambda_{\max}$ is the Lyapunov exponent

Continuous dependence on initial conditions:

$$T \to \infty \quad \text{as} \quad \Delta x \to 0$$

Lorenz (1969): $T \to T_0 < \infty \quad \text{as} \quad \Delta x \to 0$

It is proposed that certain formally deterministic fluid systems which possess many scales of motion are observationally indistinguishable from indeterministic systems; specifically, that two states of the system differing initially by a small "observational error" will evolve into two states differing as greatly as randomly chosen states of the system within a finite time interval, which cannot be lengthened by reducing the amplitude of the initial error.

(further developments for turbulent flows: Leith&Kraichnan 1972, Ruelle 1979, Eyink 1996 etc.)

Non-uniqueness and singularities

Eddy time-scale vs. wavenumber in Kolmogorov turbulence:

$$E(k) \sim k^{-5/3} \qquad \tau(k) \sim k^{-2/3}$$

Total time for error evolution from small to large scales:

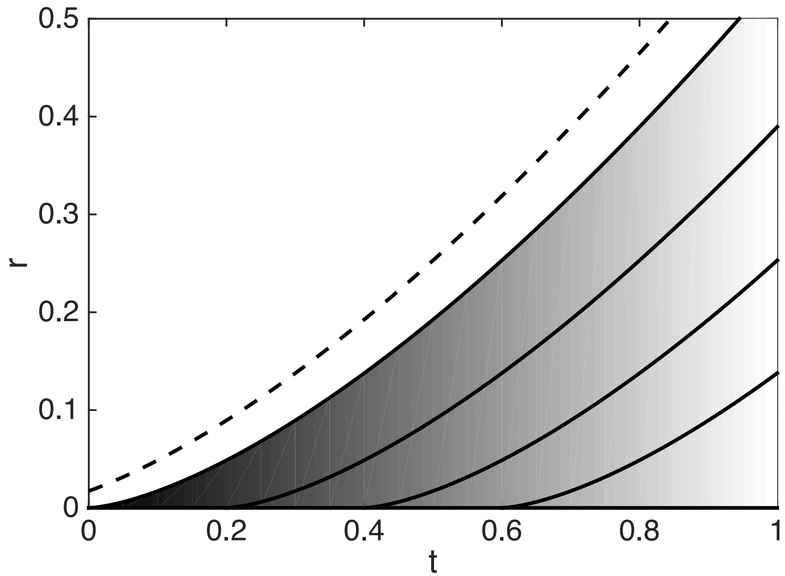
$$\tau(2^N k_L) + \cdots + \tau(2k_L) + \tau(k_L) \xrightarrow[N\to\infty]{} c\sum_{N=0}^{\infty} 2^{-2N/3} < \infty$$

Non-unique solutions of non-Lipschitz differential equations

$$\dot{r} = r^{1/3} \qquad \text{(Kolmogorov-type singularity)}$$

Solutions starting at the singularity:

$$r(t) = \begin{cases} 0 & t \le t_s; \\ \left(\frac{2(t-t_s)}{3}\right)^{3/2}, & t > t_s; \end{cases}$$



Lagrangian spontaneous stochasticity

Richardson experiments (20s) with separation of balloons:

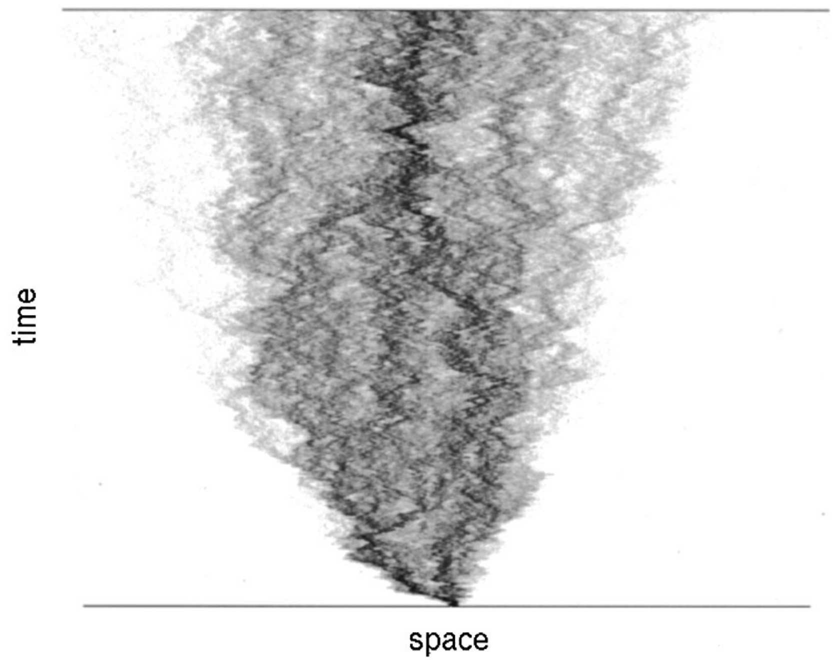
$$\rho(t)^2 \sim At^3$$

(not exponential as in deterministic chaos)

Particle diffusion (Brownian motion):

$$d\mathbf{R} = \mathbf{v}(\mathbf{R},t)\,dt + \sqrt{2\kappa}\,d\boldsymbol{\beta}(t) \qquad \kappa \to 0$$

with deterministic rough (non-Lipschitz) velocity.
Solution remains diffusive in non-diffusive limit
(spontaneous stochasticity in Lagrangian formulation).



Falkovich, Gawedzki, Vergassola 2001

Back to a full flow system
(velocity is a dynamical variable):

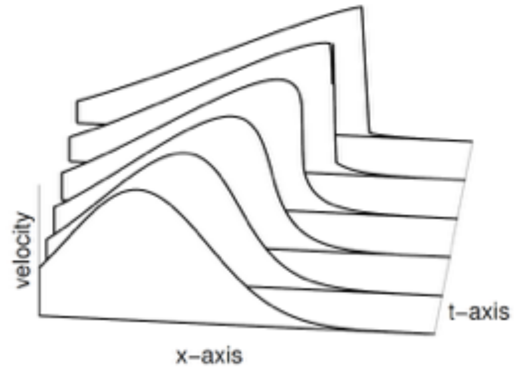
What is the origin of singularities in velocity field (for large Re limit)?

If trajectories are stochastic, why velocities are deterministic?

Burgers equation

Dynamical system view of blowup in inviscid Burgers equation

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}, \quad x, t \in \mathbb{R}, \quad \nu \to 0^+ \quad f = u^2/2.$$



Implicit solution: $u = u_0(x_0), \quad x = x_0 + (t - t_0)u$

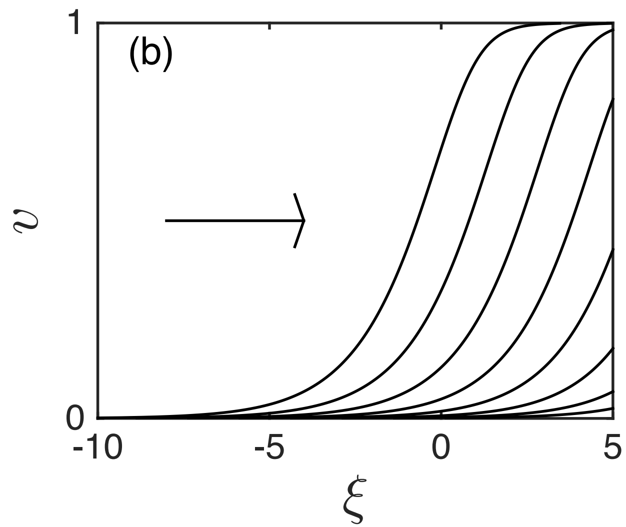
Blowup (generic solution, simplified by symmetries): $x = ut - u^3 + o(u^3)$

Renormalized (logarithmic) coordinates:

$$t = -e^{-\tau}, \quad x = e^{-\xi}, \quad u = -ve^{\tau - \xi}$$

Renormalized inviscid equation: $\frac{\partial v}{\partial \tau} = -v + v^2 - v\frac{\partial v}{\partial \xi}$

$$e^{-\xi} = ve^{-\xi} + v^3 e^{3\tau - 3\xi}$$



Renormalized solution: $v = F(\xi - a\tau), \quad a = 3/2,$

$$1 = F + e^{-2\eta} F^3$$

stable steady-state traveling wave

Dynamics after blowup

Renormalized (logarithmic) coordinates:

$$t = e^{\tau}, \quad x = e^{-\xi}, \quad u = -we^{-\tau-\xi}$$

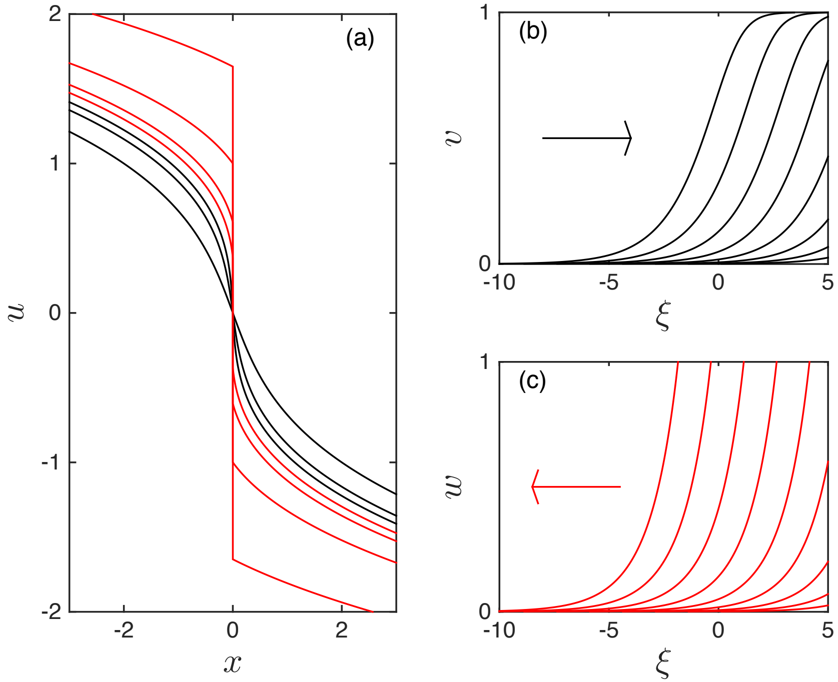
Renormalized inviscid equation:

$$\frac{\partial w}{\partial \tau} = w + w^2 - w\frac{\partial w}{\partial \xi}$$

Renormalized solution:

$$w = G(\xi + a\tau), \quad a = 3/2, \quad 1 = -G + e^{-2\eta}G^3.$$

"reflected" steady-state traveling wave
moving in opposite direction

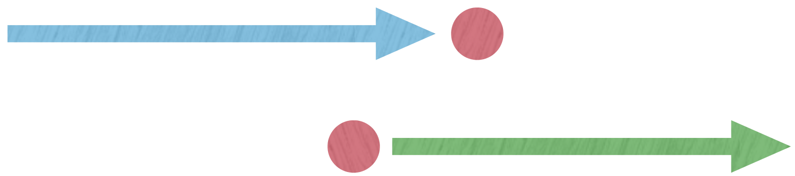


internal "clock" of blowup

Blowup: before and after

$$t = t_b$$

$$\tau = -\log(t_b - t) \quad \infty$$



$$-\infty \quad \tau = \log(t - t_b)$$

What happens if the traveling waves in renormalized system become unstable?

One can expect periodic, quasi-periodic or chaotic waves

Models

Nonlinearity and nonlocality

Incompressible Navier-Stokes equations:

$$\partial_t v_i + v_j \partial_j v_i = -\partial_i p + \nu \partial_{jj} v_i,$$
$$\partial_i v_i = 0.$$

NS equations resolved w.r.t. pressure:

$$\partial_t v_i + \left(\delta_{i\ell} - \partial_{i\ell}\nabla^{-2}\right)\partial_j\left(v_j v_\ell\right) = \nu \nabla^2 v_i$$

(nonlocal quadratic nonlinearity)

1D models that mimic a nonlocal quadratic nonlinearity:

$$\frac{\partial u}{\partial t} + \frac{\partial g}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} + f, \quad x, t \in \mathbb{R},$$

$$g(x,t) = \frac{1}{2\pi} \int\int K(y-x, z-x) u(y,t) u(z,t) \mathrm{d}y \mathrm{d}z$$

Extra conditions on the kernel function $K(y,z)$:
energy conservation, Hamiltonian structure. etc.

Example: Constantin–Lax–Majda equation $\omega_t - v_x \omega = 0, \qquad v_x = H\omega$

Special cases: Desnyansky-Novikov shell model

$$K(y,z) = -\frac{4}{(y+z)^2} - \frac{4}{(y-2z)^2} - \frac{4}{(z-2y)^2}$$

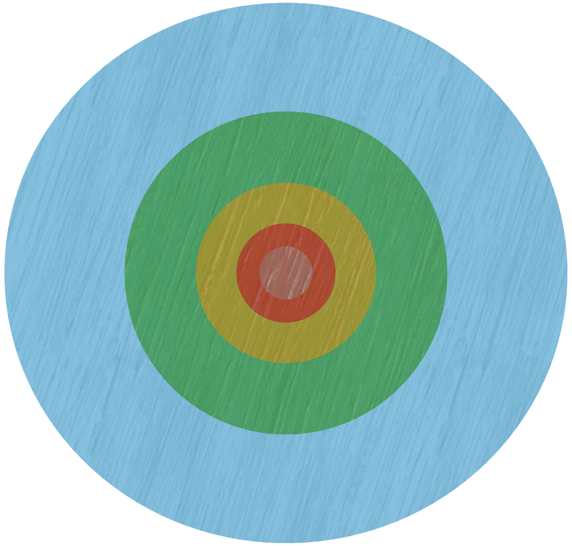
Solution representation

$$k_n = k_0\lambda^n, \quad \lambda = 2^{3/2}, \quad n \in \mathbb{Z}, \qquad 1 \leqslant k_0 < \lambda.$$

$$u_n(t) = k_n^{1/3}\hat{u}\left(k_n^{2/3}, t\right), \quad \hat{u}(k,t) = \int u(x,t)e^{-ikx}dx$$

Desnyansky-Novikov shell model equations

$$\frac{\partial u_n}{\partial t} = k_n u_{n-1}^2 - k_{n+1}u_n u_{n+1} - \nu_n u_n + f_n, \quad n \in \mathbb{Z}.$$



Shell speed	u_n
Wavenumbers	$k_n = k_0\lambda^n$
Viscosity	$\nu_n = \nu k_n^2$

Special cases: Sabra shell models

$$K(y,z) = K_\psi(y,z) + K_\psi(z,y), \quad K_\psi(y,z) = \frac{\sigma}{(\sigma y - z)^2} - \frac{(1+c)\sigma^2}{(\sigma^2 y - z)^2} - \frac{c\sigma}{(\sigma y + z)^2}$$

Sabra model equations

$$\frac{\partial u_n}{\partial t} = \mathrm{i}\left[k_{n+1}u_{n+2}u_{n+1}^* - (1+c)k_n u_{n+1}u_{n-1}^* - ck_{n-1}u_{n-1}u_{n-2}\right] - \nu_n u_n + f_n$$

$$\lambda = \sigma^{3/2} = \sqrt{2+\sqrt{5}} \approx 2.058$$

Gledzer-Ohkitani-Yamada (GOY) in 70-80th;
L'vov, Podivilov, Pomyalov, Procaccia, Vandembroucq (Sabra) in 90th

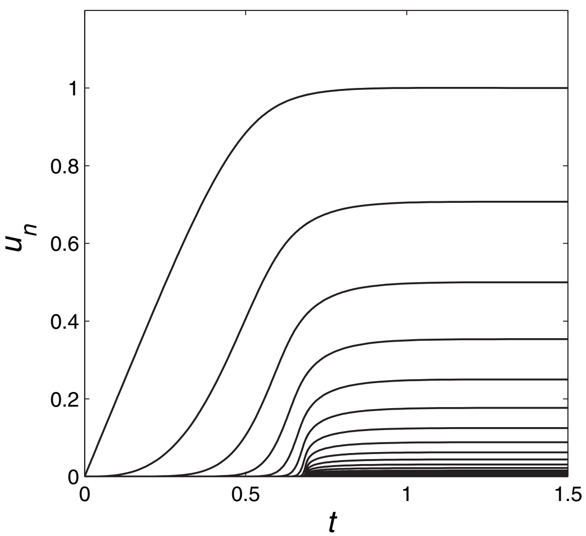
Inviscid invariants: energy, helicity, enstrophy etc. (depending on coefficients)

Dynamics in the inviscid Desnyansky-Novikov shell model: blowup to a shock wave

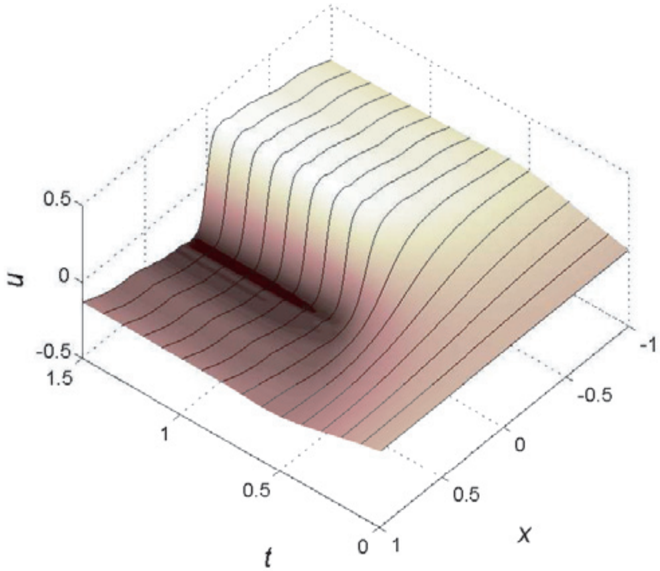
Solution blows up in finite time leading to an asymptotic stationary state with Kolmogorov scaling (for the inviscid limit)

$$u_n = k_n^{-1/3}$$

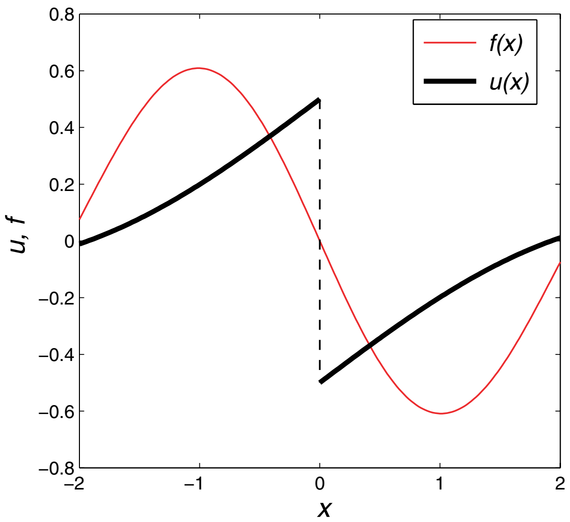
Dynamics in shell variables



Dynamics in continuous representation



Stationary state: a shock



Before blowup

Blowup in Sabra model

$$\frac{du_n}{dt} = N_n[u] - \nu k_n^2 u_n, \quad n = 1, 2, 3, \dots \qquad N_n[u] = i\left(k_{n+1}u_{n+2}u_{n+1}^* - \frac{1}{2}k_n u_{n+1}u_{n-1}^* + \frac{1}{2}k_{n-1}u_{n-1}u_{n-2}\right)$$

Renormalized system (Dombre&Gilson, 98)

$$\frac{dv_n}{d\tau} = P_n[v] - Av_n, \quad n = 1, 2, 3, \dots . \qquad A = \frac{\mathrm{Re}\sum v_n^* P_n[v]}{\sum |v_n|^2}, \quad P_n[v] = -\frac{1}{\lambda^2}v_{n+2}v_{n+1}^* + \frac{1}{2}v_{n+1}v_{n-1}^* + \frac{\lambda^2}{2}v_{n-1}v_{n-2}.$$

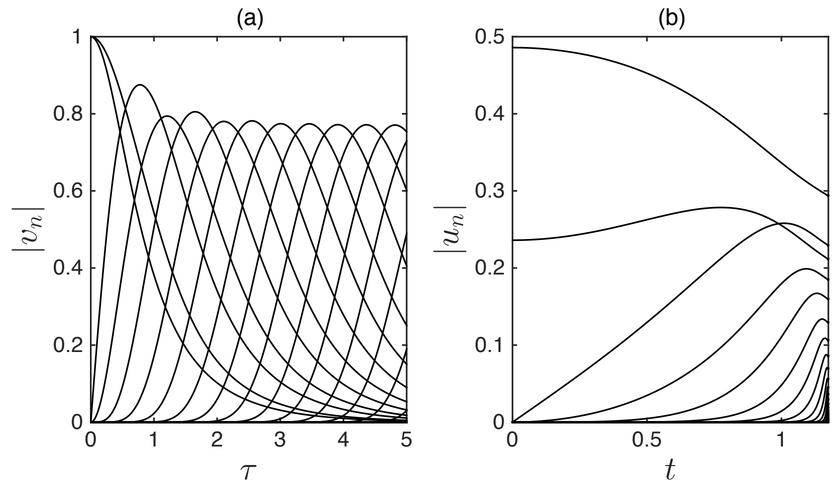
$$u_n = -ie^{\int_0^\tau A(\tau')d\tau'}k_n^{-1}v_n, \quad t = \int_0^\tau e^{-\int_0^{\tau'} A(\tau'')d\tau''}d\tau',$$

Steady-state traveling wave:

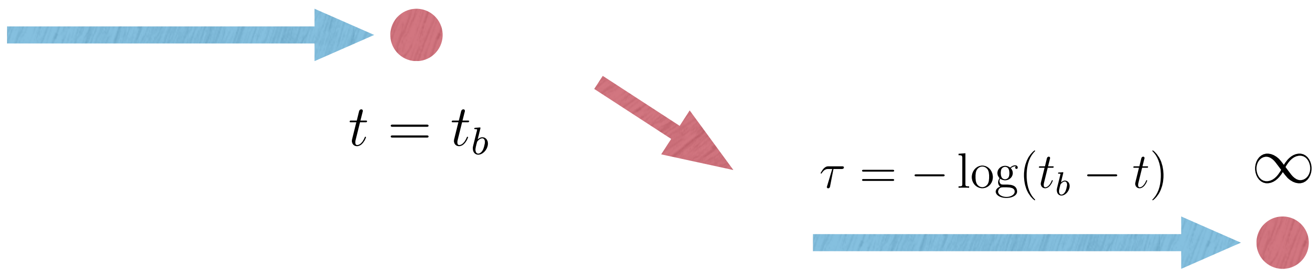
$$v_n(\tau) = e^{i\theta_n}V(n - a\tau)$$

Self-similar blowup:

$$u_n(t) = -ie^{i\theta_n}k_n^{z-1}U(k_n^z(t - t_b)), \quad t < t_b,$$



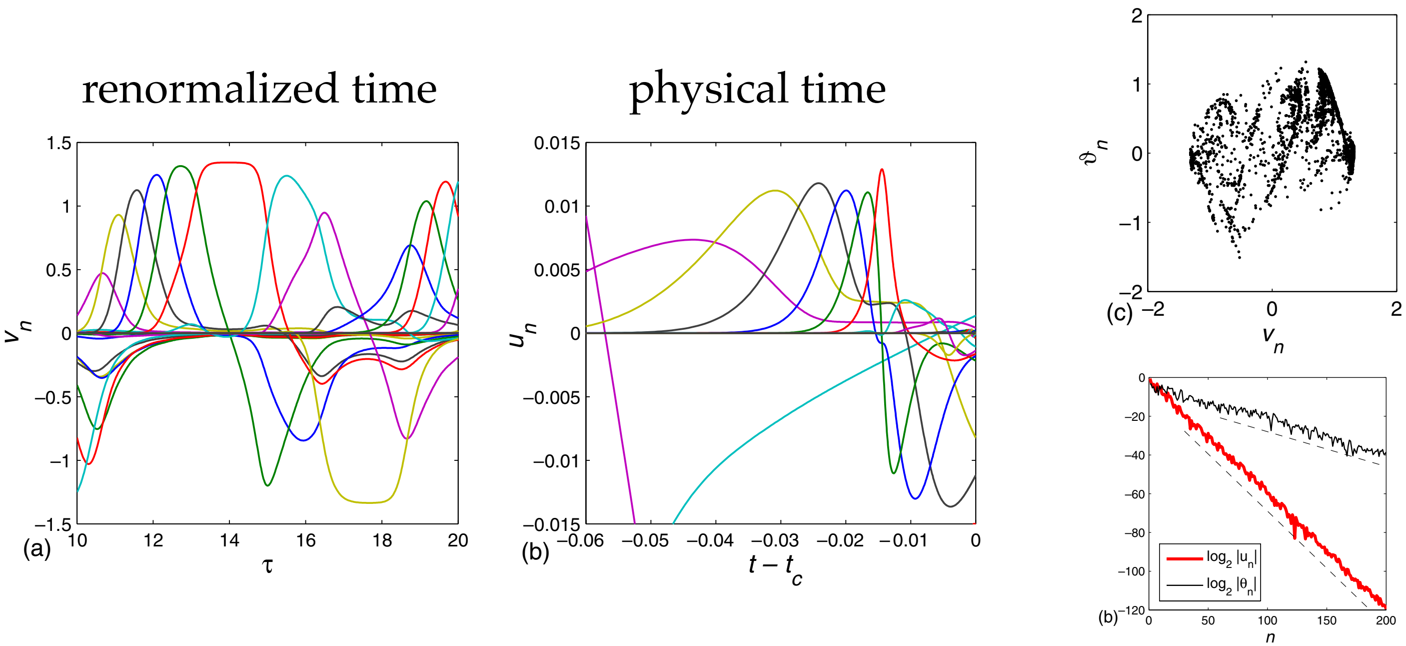
Periodic, quasi-periodic and chaotic blowup



Examples:

periodic, quasi-periodic and chaotic blowup in natural convection shell models (AM 2013);
chaotic blowup in helical shell models (de Pietro, Biferale, AM);
chaotic blowup in MHD shell models (Goedert, AM);

Chaotic blowup
in convection model:



Real-world examples?

Coalescence cascade

Is there evidence for a chaotic blowup in a real physical system?

Journal of Statistical Physics, Vol. 38, Nos. 1/2, 1985

On the Stochasticity in Relativistic Cosmology

I. M. Khalatnikov,[1] E. M. Lifshitz,[2] K. M. Khanin,[1] L. N. Shchur,[1] and Ya. G. Sinai[1]

After blowup:
non-uniqueness and
spontaneous stochasticity

Blowup state

In Sabra model:

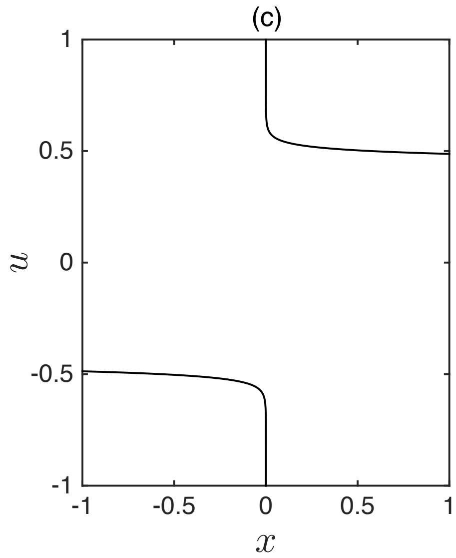
$$u_n(t_b) = -ik_n^{z-1}, \qquad z \approx 0.6975.$$

Continuous representation

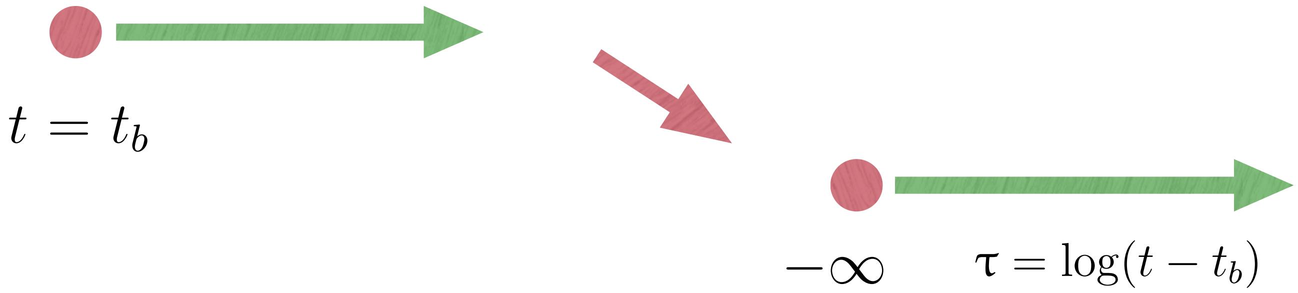
$$u_n(t) = k_n^{1/3}\hat{u}\left(k_n^{2/3}, t\right), \quad k_n = k_0\lambda^n, \qquad \hat{u}(k,t) = \int u(x,t)e^{-ikx}dx$$

$$u(x,t_b) = \frac{\Gamma(1-\beta)}{\pi}\cos\left(\frac{\beta\pi}{2}\right)|x|^{\beta-1}\mathrm{sgn}\,x$$

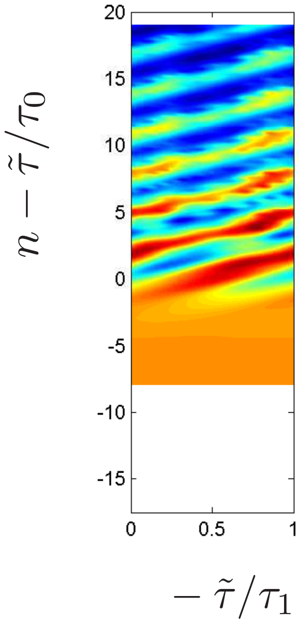
$$\beta = 2 - 3z/2 \approx 0.954$$

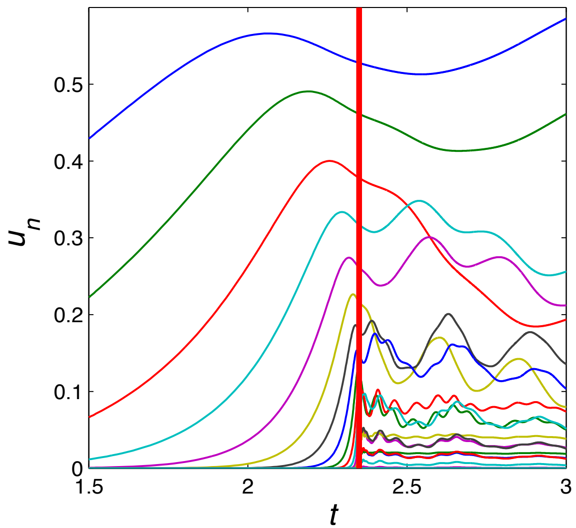


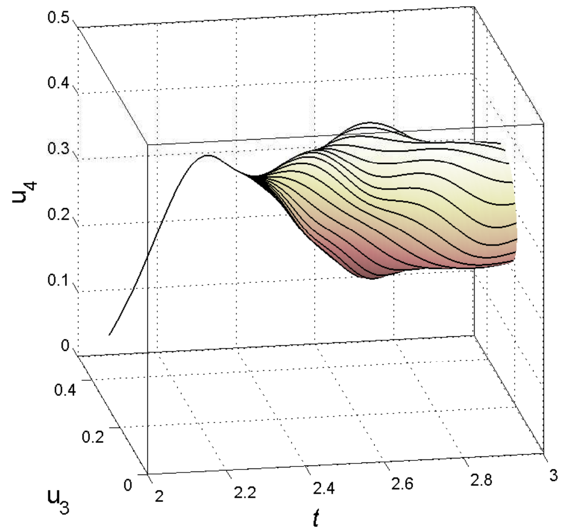
Non-stationary wave in renormalized system: unstable shock wave solution



Periodic wave in the Gledzer (real Sabra) shell model





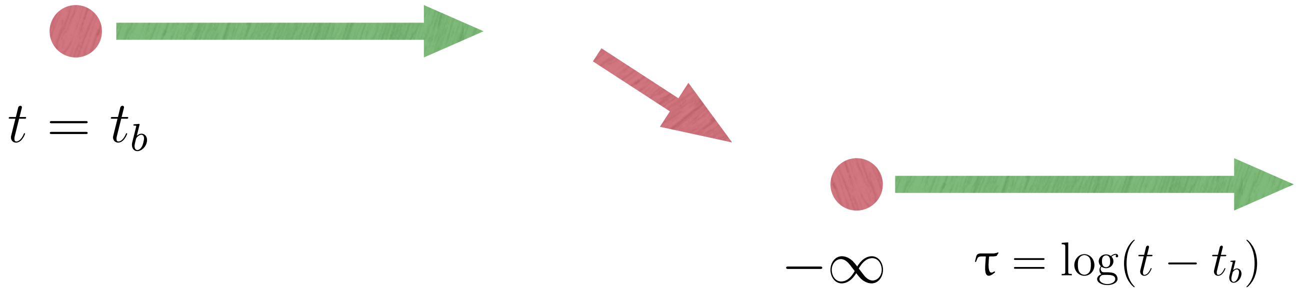


$$\nu = 2^{-4(\chi+N)} \xrightarrow[N\to\infty]{} 0$$

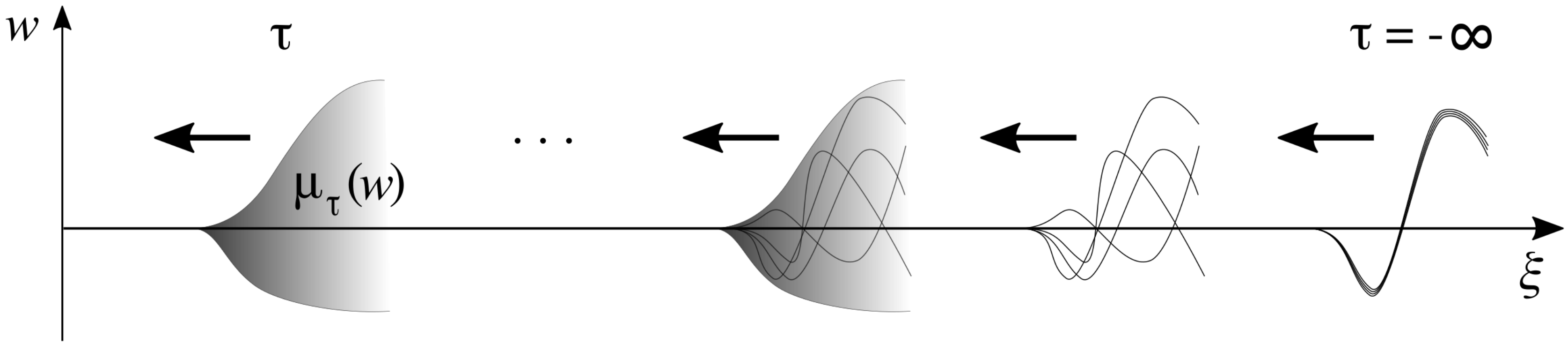
Non-unique solutions!

However, a unique solution can be chosen for a given (small) viscosity (AM 2016)

Chaotic wave in renormalized system: spontaneous stochasticity



Dynamics in renormalized time:



Implications:

- physically relevant solution is a (**spontaneous!**) probability distribution
- **unique** probabilistic description in inviscid limit
  in the form of a **steady-state traveling stochastic wave**

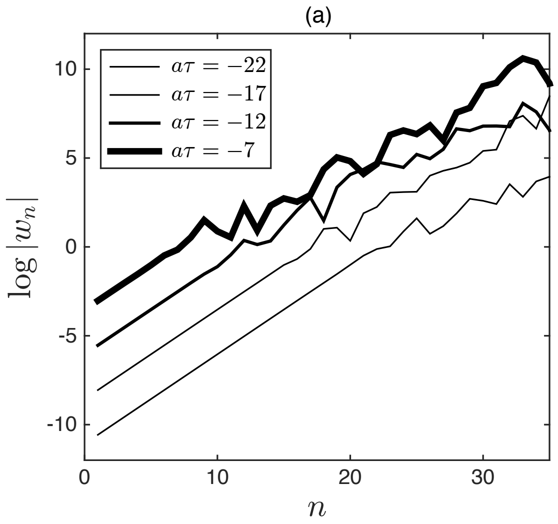
Chaotic wave after blowup in Sabra model

Renormalized system:

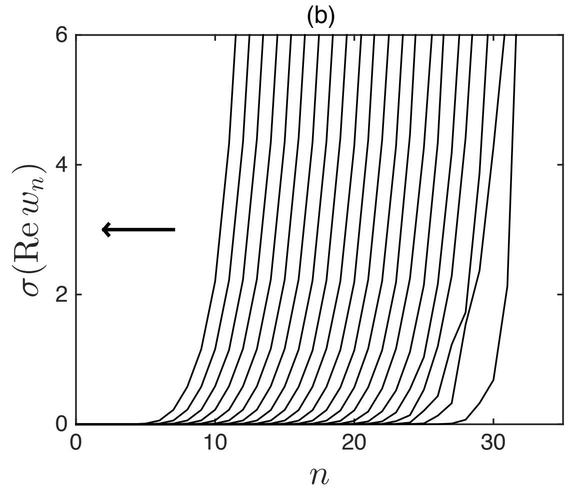
$$\frac{dw_n}{d\tau} = \left(w_n - \frac{1}{\lambda^2} w_{n+2} w_{n+1}^* + \frac{1}{2} w_{n+1} w_{n-1}^* + \frac{\lambda^2}{2} w_{n-1} w_{n-2} \right) \log \lambda.$$

$$t = t_b + \lambda^\tau, \quad u_n = -ik_n^{-1} \lambda^{-\tau} w_n = -ik_0^{-1} \lambda^{-\tau-n} w_n. \qquad \ell \propto k_n^{-1} = k_0^{-1} \lambda^{-n}$$

chaotic dynamics



stochastic wave formation



$$\nu = 10^{-15}$$

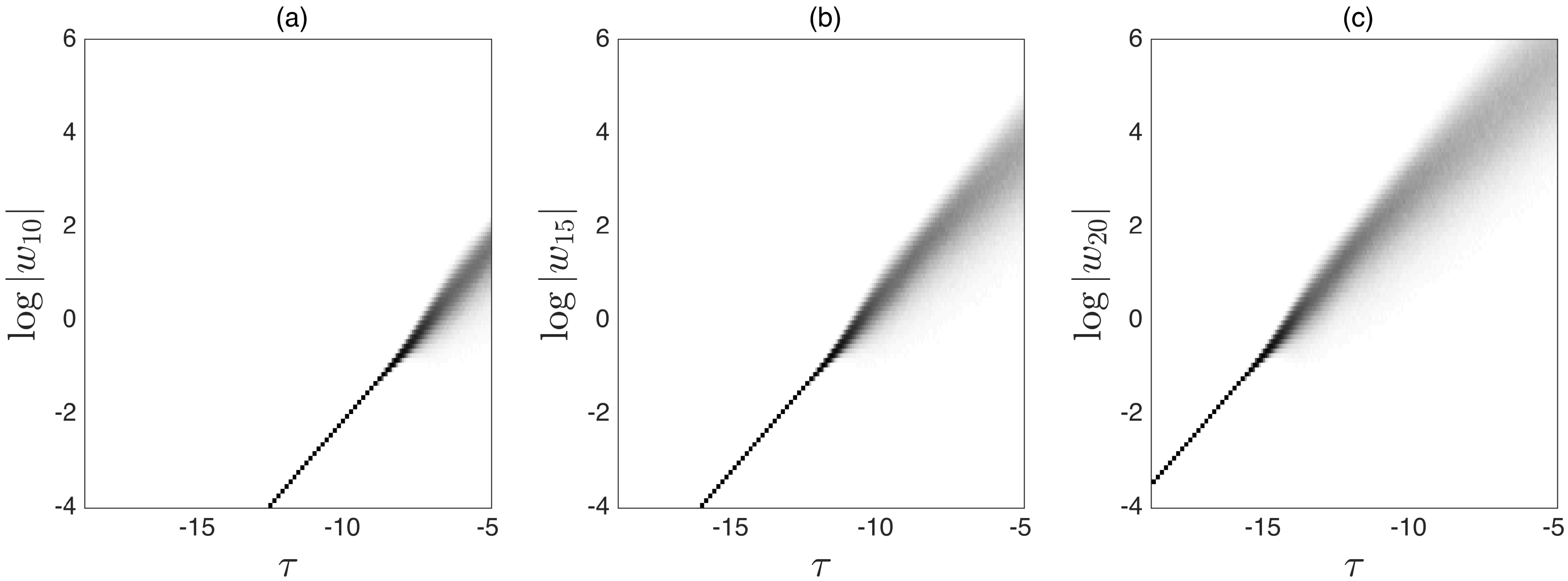
small-scale noise:

$$u_{36}(0) = (-i + 0.01x) k_{36}^{z-1}$$

Probability distribution as a steady-state traveling wave

$$\mu_{\tau+\tau_0}(w) = \mu_\tau(Tw), \qquad \tau_0 = 1/a$$

$$T : (w_1, w_2, \ldots) \mapsto (w_2, w_3, \ldots)$$



Deterministic blowup state:

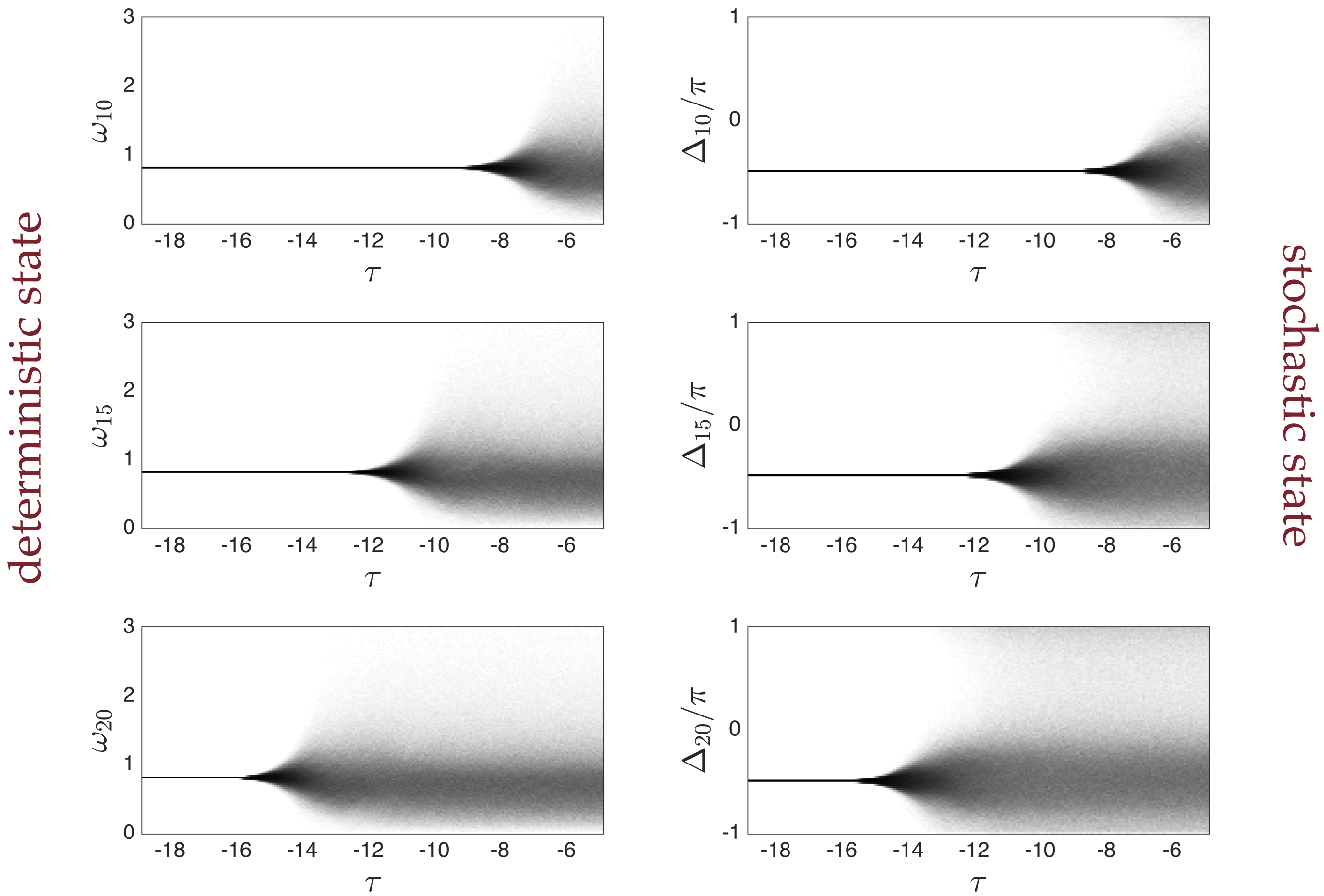
$$w_n = ik_n\lambda^\tau u_n \to k_n^z\lambda^\tau = k_0^z\lambda^{z(n+a\tau)}, \quad a = 1/z, \quad \tau \to -\infty$$

Wave speed: $\quad a = 1/z \approx 1.4337$

Traveling probability measure with constant limiting states

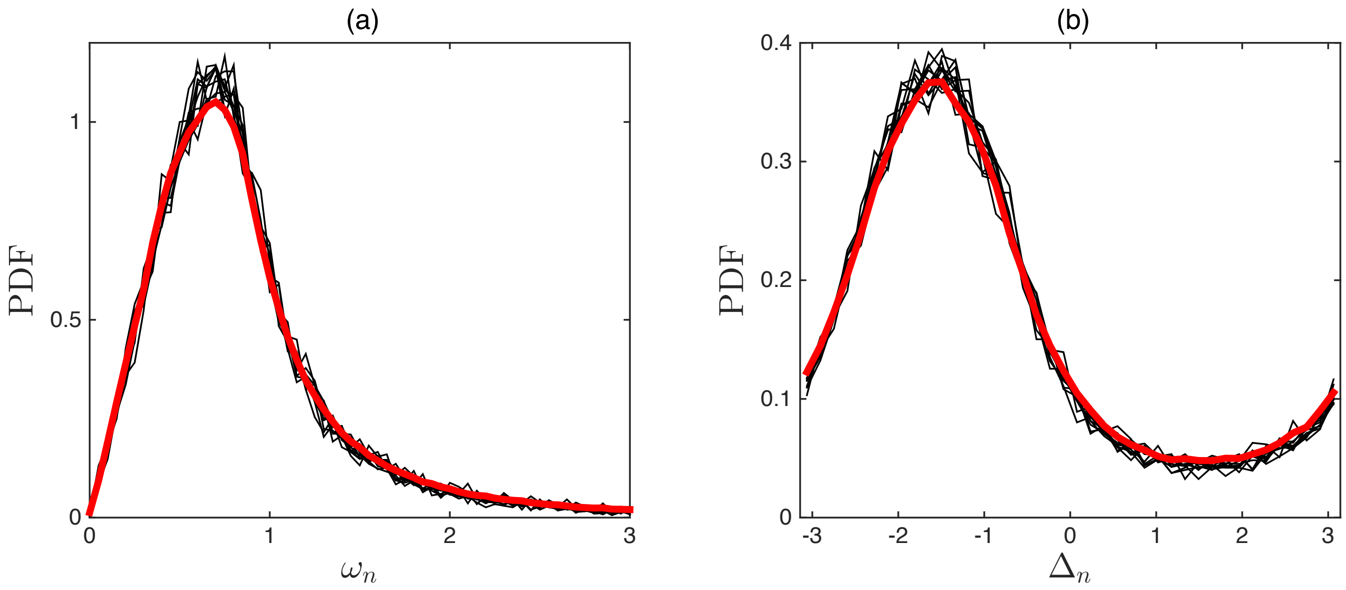
Kolmogorov hypothesis on universality of velocity increments
(Kolmogorov 62; Benzi, Biferale & Parisi 93; Eyink 2003):

$$\omega_n = |u_n/u_{n-1}|, \quad \Delta_n = \arg(u_{n-2}u_{n-1}u_n^*)$$



stable traveling wave: universal route to spontaneous stochasticity

Stochastic constant state describes the equilibrium turbulent statistics

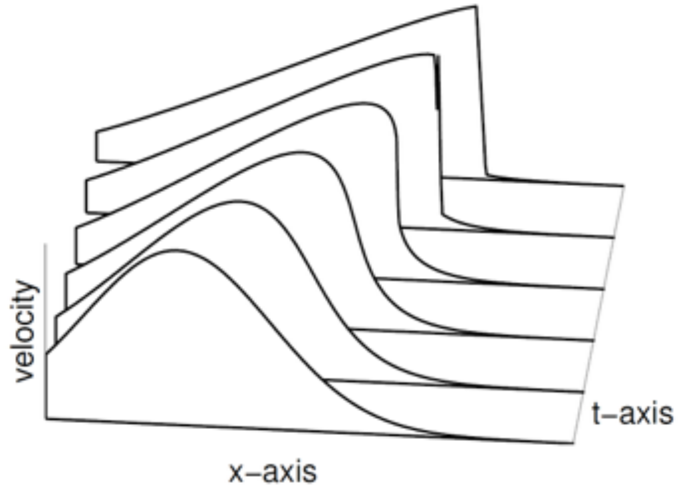


PDFs at stochastic constant state of the traveling wave
vs. PDFs of turbulent dynamics in inertial interval for the statistical equilibrium

Summary

Inviscid Burgers equation
(compressible gas dynamics)

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}, \quad x, t \in \mathbb{R}, \qquad \nu \to 0^+ \qquad f = u^2/2.$$

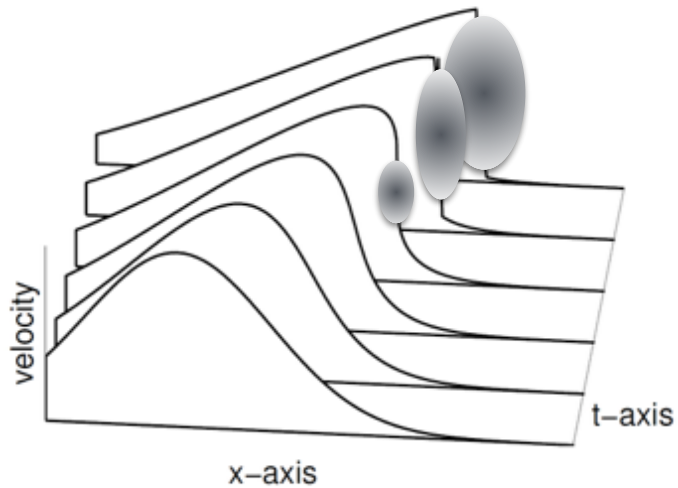


A notion of weak solution, entropy condition,
extended functional spaces, etc.

Nonlocal flux term
("incompressible" flow?)

singularity + chaos

$$f(x,t) = \frac{1}{2\pi} \int \int K(y-x, z-x) u(y,t) u(z,t) \mathrm{d}y \mathrm{d}z$$



Stochastic solutions, renormalization,
viscous regularization with infinitesimal noise, etc.

Spontaneously stochastic process as a unique solution of <u>deterministic</u> flow equations

Spontaneously stochastic solutions: general discussion

Lagrangian vs. Eulerian stochasticity

How to understand / define the inviscid (large Re) limit?

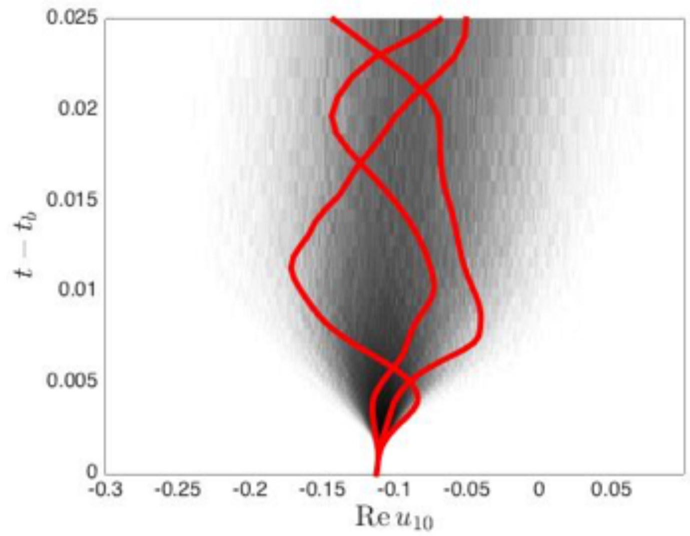
Regularization must contain an infinitesimal random term, e.g., a small-scale noise or random components in physical parameters.

Non-unique deterministic vs. unique stochastic solutions

The limiting (large Reynolds number) stochastic solution may be expected to be unique and independent of regularization.

Thus, spontaneous stochasticity is a property of inviscid **deterministic** flow equations.

Observation is a random (non-unique) realization of a unique probability measure.



Nonlinearity 29 (2016) 2238-2252